B3 - Stochastic Processes : Mid-Semester Exam

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February 21, 2025. Time : 2.00 - 4.30 PM. **Maximum points : 15**

ALL QUESTIONS CARRY 5 POINTS. ATTEMPT ANY THREE OF THEM.

2 points will be deducted if you do not write your name on the answerscript.

You are free to use any results that you have learnt in your probability courses but please cite them clearly. Provide as many details as you can.

1. For $d \geq 2$, let T_d be an infinite d-ary tree rooted at o i.e., o has d-neighbours, each of whom have d-further neighbours and so on. Edges in T_d are deleted with probability $1-p$ and retained with probability p independently of other edges. Call the subsequent graph with retained edges $T_d(p)$. Show the following:

$$\mathbb{P}\left\{\text{there exists an infinite path from } o \text{ in } T_d(p)\ \right\} = \begin{cases} 0 & \text{if } pd < 1 \\ > 0 & \text{if } pd > 1 \end{cases}$$

2. (a) Show that the Poisson random variable is sub-exponential. (**2**)

   (b) Suppose that $X_1, X_2, \ldots$ are a sequence of zero mean sub-Gaussian random variables (not necessarily independent) with variance factor ν. Show that for an absolute finite constant C,

$$\mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1 + \log i}}\right] \leq C\sqrt{\nu}. \quad (\mathbf{3})$$

3. Let G_n be the graph with vertex set $V_n = \{-n, \ldots, n\}^d \subset \mathbb{Z}^d, d \geq 1$ and edges between $x, y \in V_n$ if $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i| = 1$. Let $p \in (0, 1)$. Define $G_n(p)$ to be the random subgraph of G_n with vertex set V_n and edges in G_n are retained independently with probability p (and deleted with probability $1-p$). Let $I_d(n, p)$ denote the number of vertices of degree d in $G_n(p)$. Show that as $n \to \infty$,

$$(2n)^{-d} I_d(n, p) \to \binom{2d}{d} p^d (1 - p)^d, \quad \text{a.s.}$$

4. Let $X_1, \ldots, X_n$ be independent random elements taking values in finite sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ respectively. Let $f : \prod_{i=1}^n \mathcal{X}_i \to \mathbb{R}$ be a function and set $Y = f(X)$. Assume that $\mathbb{E}[Y^2] < \infty$. Prove *tensorization of variance* of Y using Martingales i.e., show that

$$\mathsf{VAR}(Y) \leq \sum_{i=1}^n \mathbb{E}\Big[\mathsf{VAR}\Big(Y \mid X_1, \ldots, \hat{X}_i, \ldots, X_n\Big)\Big].$$

Notions and Notation-

Conditional Variance: For random variable Y and a random element X, define $\mathsf{VAR}(Y \mid X) := \mathbb{E}\Big[\big(Y - \mathbb{E}[Y \mid X]\big)^2 \mid X\Big]$.

Sub-Gaussian random variable with variance factor ν: $\Psi_{X-\mu}(s) \leq s^2\nu/2$, for $s \in \mathbb{R}$ and where Ψ is the cumulant generating function i.e., $\Psi_X(s) = \log \mathbb{E}\big[e^{sX}\big]$.

Sub-exponential random variable with parameters ν and α: $\Psi_{X-\mu}(s) \leq s^2\nu/2$, for $|s| \leq \alpha^{-1}$.

Hölder's inequality: Let $w_j \geq 0$, $\sum_{j=1}^n w_j \leq 1$ be weights and Y_j's be non-negative random variables such that $\mathbb{E}\Big[Y_j^{1/w_j}\Big] < \infty$ for all j. Then, we have that

$$\mathbb{E}\left[\prod_{j=1}^n Y_j\right] \leq \prod_{j=1}^n \mathbb{E}\Big[Y_j^{1/w_j}\Big]^{w_j}.$$

Gamma function: $\Gamma(a) = \int_0^\infty e^{-s} s^{a-1}\, ds$ for $a > 0$ and $\Gamma(a) \leq a^a$.

Stirling's Approximation: $\sqrt{2\pi n}\ (\frac{n}{e})^n\ e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\ (\frac{n}{e})^n\ e^{1/(12n)}$.